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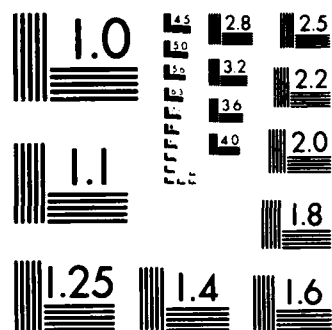
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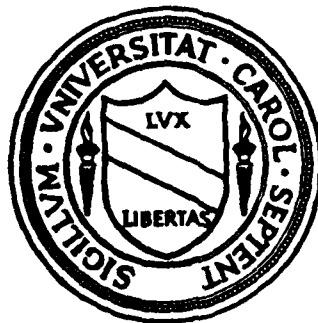
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# CENTER FOR STOCHASTIC PROCESSES

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REMARK ON THE MULTIPLE WIENER INTEGRAL

by

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# REMARK ON THE MULTIPLE WIENER INTEGRAL

by

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## Abstract

A short proof is given for Ito's result that the multiple Wiener integral can be written as an iterated stochastic integral, using the martingale property of Brownian motion and a simple property of symmetric tensor products of the  $L^2$ -space.



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## 1. Introduction

The important fact of the representability of  $L^2$ -Brownian functionals as an Ito stochastic integral may be proved by using the homogeneous chaos decomposition of the  $L^2$ -space of Brownian motion and showing as main step, that the multiple Wiener integral can be written as an iterated Ito stochastic integral (cf. G. Kallianpur [4]).

For the proof of this latter fact, usually it is referred to Ito's way (cf. [2]) to showing the representation

$$I_n(f) = n! \int_0^1 dW_{t_1} \left( \int_0^{t_1} dW_{t_2} \left( \dots \int_0^{t_{n-1}} dW_{t_n} f(t_1, \dots, t_n) \right) \right),$$

(where  $I_n$  denotes the  $n$ -fold multiple Wiener integral,  $(W_t)_{t \in [0,1]}$  Brownian motion,  $f: [0,1]^n \rightarrow \mathbb{R}$  a special elementary function of the form

$$f = \sum_{i_1, \dots, i_n=1}^p a_{i_1, \dots, i_n} 1_{A_{i_1} \times \dots \times A_{i_n}}, \quad A_1, \dots, A_p \text{ a partition of } [0,1]$$

into Borel measurable subsets,  $a_{i_1, \dots, i_n} = 0$  if two of the indices  $i_1, \dots, i_n$  are equal, and  $f_0$  the symmetrization of  $f$ )

and to extending this formula by the usual technique of approximation and the properties of multiple Wiener integral and the Ito stochastic integral.

Since it appears intricate to calculate the above iterated Ito stochastic integral for the symmetrization even of a special elementary function, I apologize the following short proof, which uses the martingale properties of Brownian motion, a well-known formula transferring different forms of Hermitian polynomials into each other, and a simple property of symmetric tensor products.

## 2. Representation of the multiple Wiener integral as an iterated Ito stochastic integral

We will use the following notations:

$\otimes^n L^2([0,1])$  the  $n$ -fold tensor product of the space  $L^2([0,1], \lambda)$ ,  $\lambda$  the Lebesgue measure on  $[0,1]$

$\otimes_o^n L^2([0,1])$  cl - lin  $\{f_1 \otimes_o \dots \otimes_o f_n : f_i \in L^2([0,1]), 1 \leq i \leq n\}$ ,  
the corresponding symmetric tensor product, where

$$f_1 \otimes_o \dots \otimes_o f_n := \frac{1}{n!} \sum_{\pi} f_{\pi_1} \otimes \dots \otimes f_{\pi_n}.$$

$\pi = (\pi_1, \dots, \pi_n)$  a permutation of the integers  $\{1, \dots, n\}$ , cl - lin  $A$  the closed linear span of the set  $A$ .

$L_o^2([0,1]^n)$  denotes the symmetric  $L^2$ -functions on  $[0,1]^n$ .

The (unnormalized)  $n$ -fold multiple Wiener integral  $I_n$  for a special elementary function  $f$  of the form

$$f = \sum_{i_1, \dots, i_n=1}^p a_{i_1, \dots, i_n} 1_{A_{i_1}} \otimes \dots \otimes 1_{A_{i_n}}$$

is given by

$$I_n(f) = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} W(A_{i_1}) \dots W(A_{i_n}),$$

where

$$W(A) = \int_0^1 1_A dW_t$$

is the stochastic measure corresponding to the Brownian motion  $(W_t)_{t \in [0,1]}$

$(A_1, \dots, A_p)$  a partition of  $[0,1]$  into Borel measurable subsets,  $a_{i_1, \dots, i_n} = 0$  if two of the indices  $i_1, \dots, i_n$  are equal).

Identifying  $\otimes_o^n L^2([0,1])$  with  $L_o^2([0,1]^n)$ , it is clear that the symmetric tensor  $\otimes^n g$  ( $g \in L^2([0,1])$ ) corresponds to  $f \in L_o^2([0,1]^n)$ ,  $f(t_1, \dots, t_n) = g(t_1)g(t_2) \dots g(t_n)$ .

Lemma 1.

$$\text{Let } h_n(x) := (-1)^n \frac{1}{(n!)^{\frac{1}{2}}} e^{\frac{x^2}{2}} \left( \frac{d}{dx} \right)^n e^{-\frac{x^2}{2}}$$

be the n-th normalized Hermite polynomial and

$$H_n(t, x) := \frac{(-t)^n}{n!} e^{\frac{x^2}{2t}} \left( \frac{d}{dx} \right)^n e^{-\frac{x^2}{2t}}$$

Then

$$H_n(t, x) = \frac{t^{\frac{n}{2}}}{n!} (n!)^{\frac{1}{2}} h_n\left(\frac{x}{t^{\frac{1}{2}}}\right)$$

This formula is well-known.

Lemma 2.

$$\otimes_o^n L^2([0, 1]) = \text{cl-lin} \{ \otimes^n g : g \in L^2([0, 1]) \}$$

Proof.

It is to show that for every  $h_1 \otimes_o \dots \otimes_o h_n \in \otimes_o^n L^2([0, 1])$  there exist  $g_1, \dots, g_r \in L^2([0, 1])$ ,  $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ ,  $r \in \mathbb{N}$  such that

$$h_1 \otimes_o \dots \otimes_o h_n = \sum_{i=1}^r \lambda_i \otimes^n g_i$$

We proceed by induction:

- i) Let  $h_1 \otimes_o \dots \otimes_o h_n$  be of the form  $h_1^k \otimes_o h_2^{n-k}$ , where  $h_i^k := \otimes^k h_i$ . Consider  $h_1^k \otimes_o h_2^{n-k}$  as tensor of the symmetric tensor algebra  $S(E)$  on the vector space  $E := \text{lin} \{h_1, h_2\}$ .  $S(E)$  is isomorphic to the polynomial algebra  $\mathbb{R}[x_1, x_2]$  over  $\mathbb{R}$  and  $h_1^k \otimes_o h_2^{n-k}$  corresponds to  $x_1^k x_2^{n-k}$ . The polynomials  $\{x_1^k x_2^{n-k} : k=0, \dots, n\}$  form a basis for the homogeneous polynomials of degree  $n$  in two variables.

$$\text{Consider } e_i^n := (a_{i1}x_1 + a_{i2}x_2)^n =$$

$$= \sum_{k=0}^n \binom{n}{k} a_{i1}^k a_{i2}^{n-k} x_1^k x_2^{n-k}.$$

The determinant of the matrix

$$\left| \binom{n}{0} a_{i1}^0 a_{i2}^n, \binom{n}{1} a_{i1}^1 a_{i2}^{n-1}, \dots, \binom{n}{n} a_{i1}^n a_{i2}^0 \right|_{i=0, \dots, n}$$

is a polynomial in  $a_{ij}$  ( $0 \leq i \leq n$ ,  $1 \leq j \leq 2$ ), which is not identically zero.



Thus we may fix suitable  $a_{ij}$ , such that this determinant is not zero, i.e. there exist linearly independent polynomials  $e_0^n, \dots, e_n^n$  of degree  $n$ , which span  $R_n[x_1, x_2]$ .

Thus there exist  $\lambda_1, \dots, \lambda_n$  with

$$x_1^k x_2^{n-k} = \sum_{i=0}^n \lambda_i e_i^n.$$

The polynomial  $e_i^n$  corresponds in  $S(E)$  to  $\otimes^n g_i$ , where  $g_i = (a_{i1}h_1 + a_{i2}h_2)$ .

- ii) Now identify  $h_1 \otimes \dots \otimes h_n$  with a tensor in  $S(E)$ ,  $E = \text{lin}\{h_1, \dots, h_n\}$ , and thus with a polynomial  $x_1^{k_1} \cdot x_2^{k_2} \dots x_n^{k_n}$ , where

$$\sum_{i=1}^n k_i = n.$$

By induction we assume

$$x_1^{k_1} (x_2^{k_2} \dots x_n^{k_n}) = \sum_{i=0}^r \lambda_i x_1^{k_1} e_i^{n-k_1}$$

The elements of the sum are polynomials of degree  $n$  in two variables (with  $e_i$  linear, homogeneous in  $x_2, \dots, x_n$ ); thus they can be represented as linear combinations of certain  $f_j^n$  (with  $f_j$  linear, homogeneous in  $x_1, \dots, x_n$ ). To  $f_j^n$  correspond  $\otimes^n g_j \in S(E)$ , which give the representation of  $h_1 \otimes \dots \otimes h_n$  we have been looking for.

#

Lemma 3.

Let  $(\Omega, F, P)$  be a probability space,  $(F_t)_{t \geq 0}$  a right-continuous filtration,  $X_0 = 0, (X_t)_{t \geq 0}$  a continuous semi-martingale.

Then

$$\int_0^t dX_{t_1} \int_0^{t_1} dX_{t_2} \dots \int_0^{t_{n-1}} dX_{t_n} = H_n(\langle X, X \rangle_t, X_t)$$

For the proof of this result, see for example [1].

Now Ito's result can be proved very easily:

Proposition (Ito [2]).

The multiple Wiener integral can be written as an iterated stochastic integral, i.e. for  $f \in L^2([0,1]^n)$ :

$$I_n(f) = n! \int_0^1 dW_{t_1} \left( \int_0^{t_1} dW_{t_2} \dots \left( \int_0^{t_{n-1}} dW_{t_n} f_0(t_1, \dots, t_n) \right) \right),$$

where  $f_0$  is the symmetrization of  $f$ .

Proof. By the linearity of stochastic integrals, the inequality

$$\left| \int_0^1 dW_{t_1} \left( \int_0^{t_1} dW_{t_2} \dots \left( \int_0^{t_{n-1}} dW_{t_n} g(t_1) \dots g(t_n) \right) \right) \right|^2 \leq \left| g \right|^{2n}$$

( $g \in L^2([0,1])$ ), and by the fact (cf. [4] and [5]) that the  $n$ -fold multiple Wiener integral induces an isometric isomorphism  $J_n$  from  $\otimes_n L^2([0,1])$  onto the  $n$ -th homogeneous chaos in  $L^2(\Omega, F^W, P)$ , such that for  $f(t_1, \dots, t_n) = g(t_1) \dots g(t_n)$ ,  $g \in L^2([0,1])$ :

$$I_n(f) = (n!)^{\frac{1}{2}} J_n(\otimes^n g) = (n!)^{\frac{1}{2}} \left| g \right|^n h_n \left( \frac{\int_0^1 g dW}{\left| g \right|} \right),$$

it suffices to show the assertion for functions  $f$  of the form  $f(t_1, \dots, t_n) = g(t_1) \dots g(t_n)$  for  $g \in L^2([0,1])$ , by Lemma 2.

Since

$$\left| g \right|^2 = \left\langle \int_0^t g(s) dW_s, \int_0^t g(s) dW_s \right\rangle, \quad (\text{cf. (1)})$$

we have by Lemma 1:

$$I_n(f) = n! H_n \left( \left\langle \int_0^t g(s) dW_s, \int_0^t g(s) dW_s \right\rangle \right),$$

and Lemma 3 proves the assertion.

#

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